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Abstract

It is commonly believed that rational explosive solutions are unstable or fragile under adaptive learning. Contrary to this belief, the paper shows that under realistic parameterizations, rational explosive solutions are both E-stable and strongly E-stable in a class of models with lagged endogenous variables. It also establishes the convergence of least squares learning process to explosive solutions. Taking a simple Cagan model of inflation as an application, the paper shows that money supply feedback rule gives rise to a rational explosive solution for prices which is learnable in real time. This provides a new potential explanation for historical high inflation. Finally, E-stability results for non-MSV explosive solutions are provided.

Keywords: money supply, learnability, inflation, E-stability

JEL Classifications: E52; E32; D83; D84

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1 Introduction

It is commonly believed that rational explosive solutions are unstable or fragile under adaptive learning; see Marcet and Sargent (1989a), Evans (1989), Evans and Honkapohja (1992, 2001). Subsequent work usually overlooks the Expectational-Stability (E-stability) properties of explosive solutions; exceptions include Branch and Evans (2011), Evans and McGough (2015). Existing literature on the E-stability of explosive solutions typically considers models without lagged endogenous variables.

In linear stochastic economic models with lagged endogenous variables, the paper firstly shows that under realistic parameterizations, the minimum state variable (MSV) explosive solution is both Expectationally-stable (E-stable) and strongly E-stable. It then establishes the convergence of least squares learning process to explosive solutions when agents learn about the growth rate of the explosive variable. As an application, the paper studies the Cagan model of inflation with a money supply feedback rule. It is shown that this rule gives rise to a rational explosive solution for price levels which is learnable. Numerical simulations illustrate the convergence of real-time learning process to the explosive solution. Finally, we provide E-stability results for MSV explosive solutions under alternative parameterizations and for non-MSV explosive solutions.

2 Model

We consider the following class of models with one expectational lead and one lag of endogenous variables studied in Evans and Honkapohja (2011, henceforth EH) p. 201-204

$$
\begin{aligned}
y_t &= \alpha + \beta E_t y_{t+1} + \delta y_{t-1} + \kappa w_t + \nu_t, && (1) \\
w_t &= \mu + \rho w_{t-1} + e_t, && (2)
\end{aligned}
$$

where $|\rho| < 1$. w_t is an exogenous AR(1) process. ν_t is an $i.i.d$ process with mean zero and constant variance.[1] Examples of models taking this form are the Lucas-Prescott model of investment under uncertainty and the Cagan model of inflation with money supply feedback rules.

To solve for MSV rational expectations (RE) equilibria, we assume that agents' PLM is

$$y_t = a + by_{t-1} + cw_t + \eta_t, \tag{3}$$

where η_t are $i.i.d.$ regression errors. Calculating conditional expectations $E_t y_{t+1}$ and substituting the expectations into (1) yield the actual law of motion (ALM)

$$y_t = T_1\left(a, b, c\right) + T_2\left(a, b, c\right) y_{t-1} + T_3\left(a, b, c\right) w_t + \frac{\nu_t}{1 - \beta b}. \tag{4}$$

The T-map which maps the coefficients in the PLM to the coefficients in the ALM is $T_1\left(a, b, c\right) = \frac{\alpha + \beta(a + c\mu)}{1 - \beta b}$, $T_2\left(a, b, c\right) = \frac{\delta}{1 - \beta b}$, and $T_3\left(a, b, c\right) = \frac{\kappa + \beta c \rho}{1 - \beta b}$. The RE solutions satisfy $T_1\left(\overline{a}, \overline{b}, \overline{c}\right) = \overline{a}$, $T_2\left(\overline{a}, \overline{b}, \overline{c}\right) = \overline{b}$, $T_3\left(\overline{a}, \overline{b}, \overline{c}\right) = \overline{c}$. The model generally has two solutions where $\overline{b} = \frac{1 \pm \sqrt{1 - 4\beta\delta}}{2\beta}$. Let $\overline{b}_+$ and $\overline{b}_-$ denote the two solutions.

3 Stability and Convergence Results

This section shows that under realistic parameterizations, the RE explosive solution of the model is both E-stable and strongly E-stable and establishes the convergence of least squares learning process to the explosive solution.

[1] Models with only one shock (i.e., either ν_t or w_t) are nested in (1)-(2). The E-stability results later do not depend on the assumption of having two shocks.

3.1 E-stability and Strong E-stability of the Explosive Solution

EH, Proposition 8.3 provided the E-stability condition of the RE solutions[2]

$$\frac{\beta}{1-\beta\bar{b}} < 1 \tag{5}$$

$$\text{and } \frac{\delta\beta}{\left(1-\beta\bar{b}\right)^2} < 1. \tag{6}$$

While EH focus on the non-explosive solutions, we instead study explosive solutions. We consider the following set of parameters:

$$0 < \beta < 1,\ \delta < 0. \tag{7}$$

Note $\bar{b}_+ > \frac{1}{\beta} > 1$ is an explosive solution.

Proposition 1

The MSV RE explosive solution $\bar{b}_+$ is E-stable if condition (7) holds.

Given (7), $\frac{\delta\beta}{\left(1-\beta\bar{b}_+\right)^2} < 0$ and hence (6) will be satisfied. Note given $\beta\bar{b}_+ > 1$, we have $\frac{\beta}{1-\beta\bar{b}_+} < 0$ and hence (5) holds. Figure 1 plots the T-map $T_2(b) = \frac{\delta}{1-\beta b}$ when $\beta = 0.99$, $\delta = -0.02$, $\alpha = 0$ and $\kappa = 0$; it is the parameterization of the numerical simulation in Section 4. The vertical dashed line is $b = \beta^{-1}$. The circled point is the $\bar{b}_+$ solution. In the interval (β^{-1}, ∞), the T-map is a hyperbola and decreasing function in coefficient b. So the $\bar{b}_+$ solution is E-stable.[3]

[2] This proposition also contains $\frac{\beta\rho}{1-\beta\bar{b}} < 1$ as the E-stability condition. Note if $\frac{\beta}{1-\beta\bar{b}} < 1$ holds, then $\frac{\beta\rho}{1-\beta\bar{b}} < 1$ also holds, given that $|\rho| < 1$. So $\frac{\beta\rho}{1-\beta\bar{b}} < 1$ is omitted here.

[3] Note the T-map is a decreasing function in the interval $\left(-\infty, \beta^{-1}\right)$, so the stationary MSV solution is also E-stable.

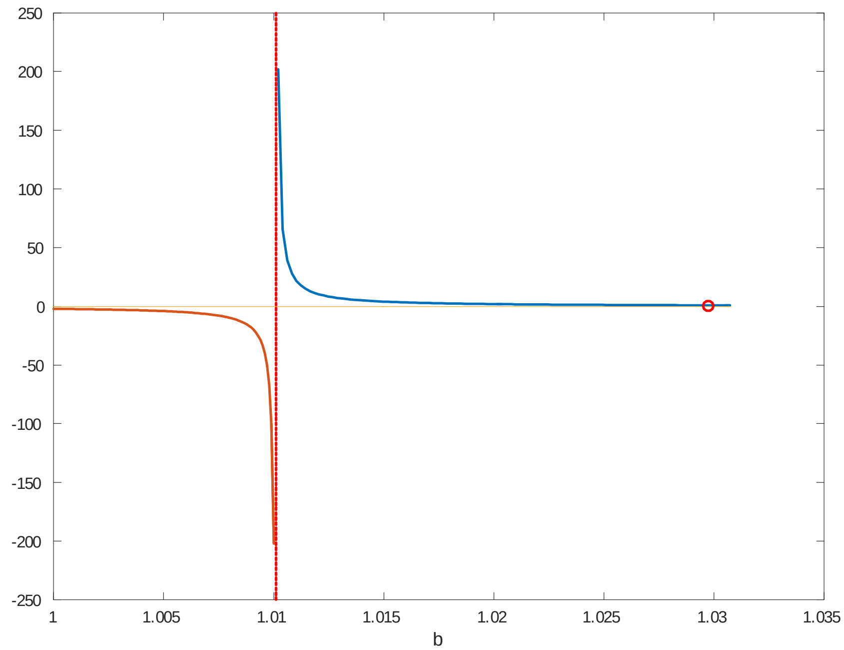

Figure 1: T-map: $T_2(b) = \frac{\delta}{1-\beta b}$.

The following proposition proves that the $\overline{b}_+$ solution is strongly E-stable and the strong E-stability condition is identical to the E-stability condition.

Proposition 2

The MSV RE explosive solution $\overline{b}_+$ is *strongly* E-stable if condition (7) holds.

Proof. Suppose agents' PLM is $y_t = a + by_{t-1} + cw_t + \sum_{i=2}^{p} d_{i-2}y_{t-i} + \sum_{i=1}^{q-1} h_{i-1}w_{t-i} + \eta_t$. We assume that the PLM is over-parameterized relative to the MSV solution and hence at least one extra lag of dependent variables or exogenous variable is added as regressors. Conditional expectations are $E_t y_{t+1} = a + by_t + c(\mu + \rho w_t) + \sum_{i=1}^{p-1} d_{i-1}y_{t-i} + \sum_{i=0}^{q-2} h_i w_{t-i}$. Substituting the expectations into model (1) yields the ALM $y_t = \frac{\alpha+\beta(a+c\mu)}{1-\beta b} + \frac{\delta+\beta d_0}{1-\beta b}y_{t-1} + \frac{\kappa+\beta c\rho+\beta h_0}{1-\beta b}w_t + \frac{\beta\sum_{i=2}^{p-1} d_{i-1}y_{t-i}}{1-\beta b} + \frac{\beta\sum_{i=0}^{q-2} h_i w_{t-i}}{1-\beta b} + \frac{1}{1-\beta b}\nu_t$. Under RE, $\overline{d}_0 = \overline{d}_1 = ... = \overline{d}_{p-2} = \overline{h}_0 = \overline{h}_1 = ... = \overline{h}_{q-2} = 0$. The convergence of coefficients $(d_1, ..., d_{p-2}, h_1, ..., h_{q-2})$ in the PLM are irrelevant for the E-stability analysis because they are mapped into zeros in the ALM. In addition, because $d_0 = h_0 = 0$ under RE, we get that the strong E-stability condition is identical to the E-stability condition. ■

3.2 Convergence of Least Squares Learning to Explosive Solution

In the $\bar{b}_+$ explosive solution, dependent variables y_t depend on the constant regressor 1, y_{t-1} and w_t. y_t grow over time but the constant regressor 1 and w_t will become negligible asymptotically relative to the explosive dependent variable. Agents may chase the trend and learn about the growth rate of y_t.[4] Suppose agents employ the following plausible PLM

$$\frac{y_t}{y_{t-1}} = b + \psi_t, \tag{8}$$

where ψ_t is $i.i.d$ regression errors with constant variance.[5] Substituting the conditional expectations (i.e., $E_t y_{t+1} = b y_t$) into (1) yields the ALM under learning

$$\frac{y_t}{y_{t-1}} = \frac{\delta}{1-\beta b} + \frac{\frac{\alpha}{1-\beta b} + \frac{\kappa}{1-\beta b} w_t + \frac{\nu_t}{1-\beta b}}{y_{t-1}}. \tag{9}$$

[4]Adam et al. (2012), Kuang and Mitra (2016) develop learning models where agents learn about the (trend) growth rates. These models can generate large fluctuations in housing markets and the business cycle and are consistent with important features of observed macroeconomic expectations.

[5]Evans and Honkapohja (1994) propose a different procedure which transforms variables to make them asymptotically stationary; this enables them to establish the convergence of least squares learning process to explosive solutions. However, to avoid singularity of moment matrix and violation of Assumption A.3 in Marcet and Sargent (1989b), they need to assume that agents perceive the variance of regression errors increasing over time at a sufficient rate. Such an assumption is not needed here as agents apply least squares to (8) where the regressor is constant 1 and the associated moment matrix is not singular.

The second term on the right hand side of (9) will vanish asymptotically. Given the PLM (8), it is optimal for agents to employ the following learning algorithm[6]

$$\widehat{b}_t = \widehat{b}_{t-1} + \frac{1}{t+N}\left(\frac{y_{t-1}}{y_{t-2}} - \widehat{b}_{t-1}\right). \tag{10}$$

Agents learn about the growth rate of y. $\widehat{b}_t$ is a simple average of past growth rates of y. As is standard in the literature, beliefs at t are updated using data up to period $t-1$. N is a measure of the precision of initial beliefs.

Proposition 3

When agents apply least squares learning algorithm (10) to the PLM (8) and if condition (7) holds, there exists a projection facility such that $\widehat{b}_t \to \overline{b}_+$ with probability one.[7]

The results of Marcet and Sargent (1989b) can be applied to establish the convergence of least squares learning. The growth rate b is mapped to $\frac{\delta}{1-\beta b}$ in the ALM (9). The convergence requires that $\frac{\delta\beta}{\left(1-\beta\overline{b}_+\right)^2} < 0$ which is satisfied given (7). Explosive outcomes are robustly obtained if agents apply least squares learning to the plausible PLM (8).

4 An Application

This section considers a simple Cagan model of inflation as in EH, p.13.[8] Price levels follow $p_t = \beta E_t p_{t+1} + \varphi m_t$ with the discount factor $1 > \beta > 0$ and $\varphi > 0$. Money supply follows a feedback rule $m_t = \overline{m} + \xi p_{t-1} + u_t$ where u_t are *i.i.d.* innovations. $\xi < 0$ says that money supply responds negatively to lagged price levels. Without loss of generality, $\overline{m}$ is set to

[6] A Bayesian micro-foundation of the learning algorithm is provided in e.g., Kuang (2014). $\widehat{b}_t$ can be interpreted as Bayesian posterior mean.

[7] As usual, a projection facility which projects agents' beliefs back to the neighborhood of the RE solution is needed to establish the result; for further details, see Marcet and Sargent (1989b), or Evans and Honkapohja (1994) or EH.

[8] Except some notations here are different from EH.

zero. This leads to the equation

$$p_t = \beta E_t p_{t+1} + \delta p_{t-1} + \nu_t, \tag{11}$$

which is a special case of (1)-(2) with $\alpha = 0, \delta = \varphi\xi < 0$, $w_t = 0$, $\nu_t = \varphi u_t$. The RE solutions are $y_t = \bar{b} y_{t-1} + \eta_t$. Proposition 1 and 2 imply the following results.

Corollary 4 *The RE explosive solution $\bar{b}_+$ for price levels in model (11) are both E-stable and strongly E-stable.*

Suppose agents learn about the growth rate of prices using (10). $\widehat{b}_t$ amounts to simple average of past inflation rates. Proposition 3 implies the following result.

Corollary 5 *If condition (7) holds, there exists a projection facility such that $\widehat{b}_t \to \bar{b}_+$ with probability one.*

The least squares learning model is simulated with the following parameterization. We set $\beta = 0.99$, $\delta = -0.02$.[9] In addition, the standard deviation of ν_t is set to 0.02% and $N = 200$.[10] Figure 2 provides a typical simulation for 1500 periods which eventually converges to the RE explosive solution. Note this simulation path does *not* invoke any projection facility. The upper panel plots the path for explosive prices and the lower panel the evolution of agents' price growth beliefs. Both agents' price growth beliefs in the PLM and actual price growth in the ALM eventually converge to $\bar{b}_+ = 1.03$ or 12% inflation per annum (when a period is interpreted as a quarter).[11]

Remark. We notice that the RE explosive solution for *inflation* is E-stable in the following simple variant of the New Keynesian (NK) model. Inflation is determined by

[9] Learnability of the RE explosive solution is obtained as long as ξ or δ is negative.

[10] The size of the *i.i.d* shocks does not matter for the convergence result. $N = 200$ can be interpreted as initial beliefs are obtained using a training sample of 200 periods.

[11] In this example, the other RE solution is $\bar{b}_- = -0.02$ and also E-stable.

$\pi_t = \beta E_t \pi_{t+1} + \varphi i_t$ where $1 > \beta > 0$ and $\varphi < 0$. And interest rates respond positively to inflation rate $i_t = \xi \pi_{t-1} + \epsilon_t$ where $\xi > 0$ and ϵ_t $i.i.d$ innovations. Combining the two equations yields $\pi_t = \beta E_t \pi_{t+1} + \delta \pi_{t-1} + \nu_t$ where $\delta = \varphi \xi < 0$ and $\nu_t = \varphi \epsilon_t$. This is a special case of model (1)-(2). Proposition 1 and and 2 imply that the explosive solution is both E-stable and strongly E-stable.

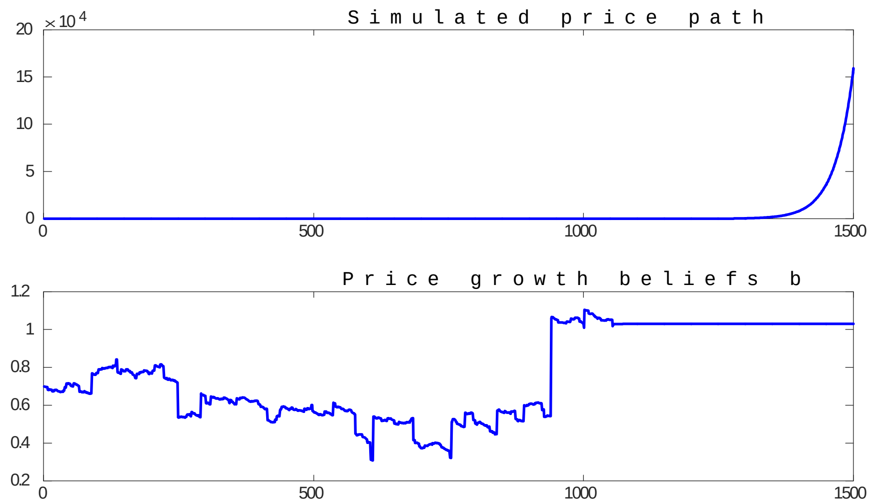

Figure 2: Convergence to the RE explosive solution for good prices

5 Further Results

This section provides E-stability results for MSV explosive solutions under alternative parameterizations and for *non-MSV* explosive solutions. Without loss of generality, we subsequently consider the model (1) with $\alpha = \kappa = 0$

$$y_t = \beta E_t y_{t+1} + \delta y_{t-1} + \nu_t. \tag{12}$$

There are generally two MSV solutions which are in the form of AR(1) process with $\bar{b}_+ = \frac{1+\sqrt{1-4\beta\delta}}{2\beta}$ and $\bar{b}_- = \frac{1-\sqrt{1-4\beta\delta}}{2\beta}$.

5.1 More on E-stability of MSV Explosive Solutions

Proposition 6 shows that even with $\delta > 0$ and $0 < \beta < 1$, explosive solutions are possibly E-stable.

Proposition 6

(a) If $0 < \beta < 1$, $\beta\delta < \frac{1}{4}$ and $\delta > 1 - \beta$, both MSV solutions are explosive. The $\overline{b}_+$ solution is E-unstable. The $\overline{b}_-$ solution is E-stable and strongly E-stable.

(b) If $0 < \beta < 1$, $\beta\delta < \frac{1}{4}$ and $0 < \delta < 1 - \beta$, the $\overline{b}_+$ solution is the unique explosive solution and E-unstable.

Proof. We outline the proof here. For part (a), note setting $\overline{b}_- > 1$ yields $\delta > 1 - \beta$. The E-stability of the $\overline{b}_-$ solution, i.e., condition (6), requires $\beta\delta < \frac{1}{4}$. Note $\overline{b}_+ > \overline{b}_- > 1$. Condition (6) is not satisfied for the solution $\overline{b}_+$. The strong E-stability of the $\overline{b}_-$ solution can easily be established by following the argument made in the proof of Proposition 2. For part (b), it can be easily shown that $0 < \overline{b}_- < 1$ and $\overline{b}_+ > 1$. And condition (6) is not satisfied for the $\overline{b}_+$ solution. ∎

A numerical example of part (a) is that $\beta = 0.3$ and $\delta = 0.71$. So $\overline{b}_+ = 2.308$ and $\overline{b}_- = 1.026$. The $\overline{b}_+$ solution is E-unstable and the $\overline{b}_-$ solution is E-stable. We note there is no simple connection between equilibrium uniqueness and E-stability of explosive solutions.

Turning to the model (12) with negative expectations feedback, i.e., $\beta < 0$. An example is the Lucas-Prescott model of investment under uncertainty; see EH, p. 201-202.

Proposition 7

If $\beta < 0$ and $\delta > 1 - \beta$, there exists two explosive solutions and both are E-stable and strongly E-stable.

Proof. The E-stability condition for both solutions is (6). First, let $\overline{b}_-$ be an explosive solution that $\overline{b}_- > 1$. This is equivalent to $\delta > 1 - \beta$. With $\beta < 0$ and $\delta > 1 - \beta$, condition

(6) is clearly satisfied. Hence, the RE explosive solution $\bar{b}_-$ is E-stable. Second, let $\bar{b}_+$ also be an explosive solution that $\bar{b}_+ < -1$. It can be easily show that this is equivalent to

$$\sqrt{1 - 4\beta\delta} > -2\beta - 1. \tag{13}$$

If $-\frac{1}{2} < \beta < 0$, then $-2\beta - 1 < 0$ and (13) is satisfied. If $\beta \leq -\frac{1}{2}$, then (13) is equivalent to

$$\delta > -1 - \beta. \tag{14}$$

Given $\delta > 1 - \beta$, (14) holds. Therefore, if $\beta < 0$ and $\delta > 1 - \beta$, $\bar{b}_+$ is an explosive solution that $\bar{b}_+ < -1$. In addition, with $\beta < 0$ and $\delta > 1 - \beta$, condition (6) is clearly satisfied. The $\bar{b}_+$ solution is E-stable. Due to space limit, we omit the proof for the strong E-stability of both explosive solutions which can easily be established by following the argument made in the proof of Proposition 2. ■

5.2 E-stability of Non-MSV Explosive Solutions

The model (12) has a class of non-MSV explosive solutions

$$y_t = \beta^{-1} y_{t-1} - \beta^{-1}\delta y_{t-2} + \xi_t - \beta^{-1}\nu_{t-1}, \tag{15}$$

where ξ_t is a martingale difference sequence. This section proves the following proposition.

Proposition 8

The class of non-MSV RE explosive solutions (15) is E-unstable.

Proof. Consider agents' PLM

$$y_t = e y_{t-1} + g y_{t-2} + f \nu_{t-1} + \eta_t, \tag{16}$$

where η_t are $i.i.d$ regression errors. Conditional expectations are $E_t y_{t+1} = ey_t + gy_{t-1} + f\nu_t$. Substituting expectations into model (12) yields the ALM

$$y_t = \frac{\delta + \beta g}{1 - \beta e} y_{t-1} + \frac{1 + \beta f}{1 - \beta e} \nu_t. \tag{17}$$

when $e \neq \beta^{-1}$. Note the PLM (16) is over-parameterized relative to the ALM (17). The latter is an AR(1) process and the coefficients on y_{t-2} and ν_{t-1} are zeros (in the ALM). The class of explosive solutions is not nested in (17) and the coefficients g and f in the PLM (16) will converge to 0. Therefore, the class of explosive solutions is E-unstable. ■

In the model with time$-t$ dating and one expectation terms, non-MSV solutions are E-unstable because these solutions are over-parameterized relative to the ALM and not nested in the ALM in the neighborhood of the explosive solutions (e.g., in the neighborhood of β^{-1} for e in this model). However, we conjecture that under certain conditions, *non-MSV* explosive solutions can be E-stable in models with time$-t-1$ dating and lagged endogenous variables, such as the model studied in Evans and Honkapohja (1994). This is because with time$-t-1$ dating and assuming agents adopt a PLM which nests non-MSV explosive solutions, the corresponding ALM can have identical functional form as the PLM (and not be over-parameterized relative to the PLM).

6 Conclusion

In models with lagged endogenous variables, the paper shows that under realistic parameterizations, rational MSV explosive solutions are both E-stable and strongly E-stable. It also establishes the convergence of least squares learning process to explosive solutions. Taking a simple Cagan model of inflation as an application, the paper demonstrates that money supply feedback rules have an undesirable property that it can give rise to a rational explosive solution for price levels which is learnable. This provides a potential new explanation for historical big inflations, such as the Great inflation in the US during the 1970s.

The paper suggests that the E-stability properties of rational explosive solutions should be carefully studied, particularly in models with lagged endogenous variables. Based on the results here, we conjecture, for example, that in the basic three-equation New Keynesian models and under realistic parameterizations, history-dependent simple interest rate rules (i.e., with responding to lagged inflation and output gaps) will give rise to rational explosive solution for *inflation* which is E-stable under learning. Prudent monetary policies should and can avoid such excessive fluctuations associated with learnable explosive solutions. We leave this issue of monetary policy design for future work.

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